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Delay-dependent Stability Criteria of the Uncertain Linear System with Multiple Time-varying States*

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Abstract: This paper deals with the problem of delay-dependent stability criteria for the uncertain linear system with multiple time-varying delays by virtue of the integral-equality. The integral equality is an improved version of the usual integral inequality and is constructed with free weighting matrices. The free terms are chosen according to the derivative of the Lyapunov-Krasovskii functional. The new improved criteria are much less conservative and more general than some existing results.

Keywords: Lyapunov-Krasovskii functional; stability; delay; integral-equality; linear matrix inequality (LMI)

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1 Introduction

Time-delays are frequently encountered in many fields of science and engineering, including communication network, manufacturing system, biology, economy and other areas. During the last decade, the problem of deriving delay-dependent stability criteria for linear time delay systems has attracted the attention of many researchers[1-15]. For systems with time-varying delay, fixed model transformations are the main methods to deal with delay-dependent stability problems[4], were used to estimate the upper of cross terms. Recently, in order to reduce the conservatism, the free-weighting matrix method was proposed in [10,12,14] to investigate the delay-dependent stability, in which the bounding techniques on some cross product terms are not involved. It is the key point and main difference that how to deal with the upper bound of the derivative of Lyapunov functional for system with time-varying delay in [4,9,10,12,14].

There are some literature for stability criteria of system with multiple time-varying delays, Fridman discussed two time-varying delays problem in [2], and Yan *et al.* discussed multiple time-varying delays problem in [13], which has given asymptotically stable condition in terms of multiple linear matrix inequalities (LMIs). Moreover, our stability criteria only need one LMI.

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In this paper, we propose some new delay-dependent stability criteria for the uncertain system with multiple time-varying delays in terms of LMIs. Compare with other stability criteria, our results overcome some main sources of conservatism. The new criterion has its own advantages, it does not use the inequality to estimate the upper bound of

$$-\int_{t-\tau_i(t)}^{t} \dot{x}^T(t)Z_i\dot{x}(t)\mathrm{d}s, \quad i=1,2,\cdots,K.$$

Notation R^n denotes the n-dimensional Euclidean space, $R^{n\times m}$ is the set of $n\times m$ real matrices, I is the identity matrix of appropriate dimensions, the notation $X>0$ (respectively, $X\geq 0$) for $X\in R^{n\times n}$ means that the matrix X is real positive definite (respectively, positive semi-definite). For an arbitrary matrix B and two symmetric matrices A and C,

$$\begin{bmatrix} A & B \\ * & C \end{bmatrix}$$

denotes a symmetric matrix, where $*$ denotes a block matrix entry implied by symmetry.

2 System description and preliminaries

Consider the following linear system with multiple time-varying delays

$$\begin{cases} \dot{x}(t) = Ax(t) + \sum\limits_{i=1}^{K} A_i x(t-\tau_i(t)), \\ x(t) = \phi(t), \quad t\in[-\tau,0], \end{cases} \tag{1}$$

where $x(t)\in R^n$ is the state vector, A, A_i $(i=1,2,\cdots,K)$ are known real constant matrices with appropriate dimensions, $\phi(t)$ is the initial condition of the system, $\tau_i(t)$ $(i=1,2,\cdots,K)$ denote time-varying continuous functions, and are assumed to satisfy (2) or (3):

$$0\leq\tau_i(t)\leq\tau_i\leq\tau, \quad \dot{\tau}_i(t)\leq\mu_i\leq\mu, \quad i=1,2,\cdots,K, \quad \forall\, t>0, \tag{2}$$

$$0\leq\tau_i(t)\leq\tau_i\leq\tau, \quad i=1,2,\cdots,K, \quad \forall\, t>0, \tag{3}$$

where τ_i and μ_i are constants. τ and μ are the upper bound of τ_i and μ_i, namely,

$$\tau=\max_{1\leq i\leq K}\{\tau_i\}, \quad \mu=\max_{1\leq i\leq K}\{\mu_i\}.$$

For any delay satisfying (2) or (3), our objective of this study is to develop new delay-dependent stability criteria which guarantee that system (1) is asymptotically stable and the system (1) subject to some uncertainties is robustly stable. For this purpose, the following lemma is introduced.

Lemma 1 Given $d(t)>0$, for any appropriately dimensioned matrices U, V, W, functions $g(s)$ and $\eta(t)$, the following equation holds

$$\begin{aligned} -\int_{t-d(t)}^{t} g^T(s)Wg(s)\mathrm{d}s &= 2\eta^T(t)V\int_{t-d(t)}^{t} g(s)\mathrm{d}s + d(t)\eta^T(t)U\eta(t) \\ &\quad -\int_{t-d(t)}^{t}\begin{bmatrix}\eta^T(t) & g^T(s)\end{bmatrix}\begin{bmatrix} U & V \\ * & W\end{bmatrix}\begin{bmatrix}\eta(t) \\ g(s)\end{bmatrix}\mathrm{d}s. \end{aligned} \tag{4}$$

Proof The following two equations hold

$$\eta^T(t)V\int_{t-d(t)}^{t} g(s)\mathrm{d}s = \int_{t-d(t)}^{t} g^T(s)V^T\eta(t)\mathrm{d}s$$

and

$$\int_{t-d(t)}^{t} \eta^T(t)U\eta(t)\mathrm{d}s = d(t)\eta^T(t)U\eta(t),$$

it is clear that Lemma 1 is true. This completes the proof.

3 New stability criteria

First, we propose delay-dependent stability criteria of the nominal system (1) satisfying (2) or (3), which are obtained through the new integral-equality and an appropriate type of Lyapunov functions.

Theorem 1 For given constants τ_i, μ_i $(i = 1,\cdots,K)$, the nominal system (1) satisfying (2) is asymptotically stable if there exist matrices $P > 0$, $Z_i = Z_i^T > 0$, $Q_i = Q_i^T > 0$, N_i $(i = 1,\cdots,K)$ and M of appropriate dimensions such that the following LMI holds

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix} < 0, \tag{5}$$

where

$$\Xi_{11} = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \Phi_{0K} & \Phi_{0K+1} \\ * & \Phi_{11} & \cdots & \Phi_{1K} & \Phi_{1K+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & \Phi_{KK} & \Phi_{KK+1} \\ * & * & \cdots & * & \Phi_{K+1K+1} \end{bmatrix},$$

and

$$\Phi_{00} = \sum_{i=1}^{K} Q_i + \sum_{i=1}^{K}(N_{i0}^T + N_{i0}) + M_0A + A^TM_0^T, \qquad \Phi_{01} = \sum_{i=1}^{K} N_{i1}^T - N_{10} + M_0A_1 + A^TM_1^T,$$

$$\Phi_{0K} = \sum_{i=1}^{K} N_{iK}^T - N_{K0} + M_0A_K + A^TM_K^T, \qquad \Phi_{0K+1} = P + \sum_{i=1}^{K} N_{iK+1}^T - M_0 + A^TM_{K+1}^T,$$

$$\Phi_{11} = -(1-\mu_1)Q_1 - N_{11}^T - N_{11} + M_1A_1 + A_1^TM_1^T, \quad \Phi_{1K} = -N_{1K}^T - N_{K1} + M_1A_K + A_1^TM_K^T,$$

$$\Phi_{1K+1} = -N_{1K+1}^T - M_1 + A_1^TM_{K+1}^T,$$

$$\Phi_{KK} = -(1-\mu_K)Q_K - N_{KK}^T - N_{KK} + M_KA_K + A_K^TM_K^T,$$

$$\Phi_{KK+1} = -N_{KK+1}^T - M_K + A_K^TM_{K+1}^T, \qquad \Phi_{K+1K+1} = \sum_{i=1}^{K} \tau_iZ_i - M_{K+1} - M_{K+1}^T,$$

$$\Xi_{12} = (\tau_1N_1, \tau_2N_2, \cdots, \tau_KN_K), \qquad \Xi_{22} = \mathrm{diag}(-\tau_1Z_1, -\tau_2Z_2, \cdots, -\tau_KZ_K),$$

$$N_i^T = (N_{i0}^T, N_{i1}^T, \cdots, N_{iK}^T, N_{iK+1}^T), \qquad M^T = (M_0^T, M_1^T, \cdots, M_K^T, M_{K+1}^T).$$

Proof We choose the Lyapunov-Krasovskii functional candidate as follows

$$
\begin{aligned}
V(x_t) &= x^T(t)Px(t)+\sum_{i=1}^{K}\Big\{\int_{-\tau_i}^{0}\int_{t+s}^{t}\dot{x}^T(v)Z_i\dot{x}(v)\mathrm{d}v\mathrm{d}s\Big\}\\
&\quad+\sum_{i=1}^{K}\Big\{\int_{t-\tau_i(t)}^{t}x^T(s)Q_ix(s)\mathrm{d}s\Big\},
\end{aligned}
$$

where $P>0$, $Z_i>0$, $Q_i>0$ $(i=1,\cdots,K)$ are to be determined. Taking the derivative of $V(x_t)$ with respect to t along the trajectory of (1) yields

$$
\begin{aligned}
\dot{V}(x_t) &\le x^T(t)(P+P^T)\dot{x}(t)+\sum_{i=1}^{K}\tau_i\dot{x}^T(t)Z_i\dot{x}(t)-\sum_{i=1}^{K}\int_{t-\tau_i}^{t}\dot{x}^T(s)Z_i\dot{x}(s)\mathrm{d}s\\
&\quad+\sum_{i=1}^{K}x^T(t)Q_ix(t)-\sum_{i=1}^{K}(1-\mu_i)x^T(t-\tau_i(t))Q_ix(t-\tau_i(t)).
\end{aligned}
\tag{6}
$$

It is easy to show

$$
-\sum_{i=1}^{K}\int_{t-\tau_i}^{t}\dot{x}^T(s)Z_i\dot{x}(s)\mathrm{d}s\le-\sum_{i=1}^{K}\int_{t-\tau_i(t)}^{t}\dot{x}^T(s)Z_i\dot{x}(s)\mathrm{d}s. \tag{7}
$$

Applying the Lemma 1, the following equality holds

$$
\begin{aligned}
&-\sum_{i=1}^{K}\int_{t-\tau_i(t)}^{t}\dot{x}^T(s)Z_i\dot{x}(s)\mathrm{d}s\\
&=\sum_{i=1}^{K}\Bigg\{2\xi^T(t)N_i\int_{t-\tau_i(t)}^{t}\dot{x}(s)\mathrm{d}s+\tau_i(t)\xi^T(t)X_i\xi(t)\\
&\quad-\int_{t-\tau_i(t)}^{t}\begin{bmatrix}\xi^T(t) & \dot{x}^T(s)\end{bmatrix}\begin{bmatrix}X_i & N_i\\ N_i^T & Z_i\end{bmatrix}\begin{bmatrix}\xi(t)\\ \dot{x}(s)\end{bmatrix}\mathrm{d}s\Bigg\},
\end{aligned}
\tag{8}
$$

where $\xi^T(t)=(x^T(t),x^T(t-\tau_1(t)),\cdots,x^T(t-\tau_K(t)),\dot{x}^T(t))$, and X_i $(i=1,\cdots,K)$ are some matrices of appropriate dimensions.

According to the Newton-Leibniz formula, we obtain

$$
x(t-\tau_i(t))=x(t)-\int_{t-\tau_i(t)}^{t}\dot{x}(s)\mathrm{d}s.
$$

Then, for matrices N_i $(i=1,\cdots,K)$ with appropriate dimensions, the following holds

$$
\sum_{i=1}^{K}2\xi^T(t)N_i\Big[x(t)-\int_{t-\tau_i(t)}^{t}\dot{x}(s)\mathrm{d}s-x(t-\tau_i(t))\Big]=0. \tag{9}
$$

On the other hand, for matrices M with appropriate dimensions, the following holds

$$
2\xi^T(t)M\Big[Ax(t)+\sum_{i=1}^{K}A_ix(t-\tau_i(t))-\dot{x}(t)\Big]=0. \tag{10}
$$

Combine (6)-(10), we can obtain

$$\dot{V}(x_t) \le \xi^T(t)\Big[\Xi_{11} + \sum_{i=1}^{K} \tau_i X_i\Big]\xi(t)$$

$$-\sum_{i=1}^{K}\int_{t-\tau_i(t)}^{t} \begin{bmatrix} \xi^T(t) & \dot{x}^T(s) \end{bmatrix} \begin{bmatrix} X_i & N_i \\ N_i^T & Z_i \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{x}(s) \end{bmatrix} \mathrm{d}s. \qquad (11)$$

When $X_i = N_i Z_i^{-1} N_i^T$, we can assure

$$\begin{bmatrix} X_i & N_i \\ N_i^T & Z_i \end{bmatrix} \ge 0.$$

Applying the Schur complement, if (5) is satisfied, then $\dot{V}(x_t) < 0$, for any $\xi(t) \neq 0$, thus the nominal system (1) is asymptotically stable. This completes the proof.

When the restriction on the derivative of the time-varying delays is removed, that is, when fast time-varying delays are allowed, choose $Q_i \equiv 0$ $(i = 1, \cdots, K)$, as described in the proof of Theorem 1, then delay-derivative-free stability condition for the nominal system (1) would follow.

Corollary 1 For given constants τ_i $(i = 1, \cdots, K)$, the nominal system (1) satisfying (3) is asymptotically stable if there exist matrices $P > 0$, $Z_i = Z_i^T > 0$, N_i $(i = 1, \cdots, K)$ and M of appropriate dimensions such that the LMI (5) holds with $Q_i \equiv 0\,(i = 1, \cdots, K)$.

Next we address the linear norm-bounded uncertainties. Suppose that matrices A and A_i have parameter perturbations as $\Delta A(t)$ and $\Delta A_i(t)$, which are in the form of

$$\Delta A(t) = DF(t)E, \quad \Delta A_i(t) = D_i F_i(t) E_i, \quad i = 1, \cdots, K, \qquad (12)$$

where D, E, D_i, E_i are known real constant matrices of appropriate dimensions and $F(t)$, $F_i(t)$ are unknown matrices functions with Lesbesgue measurable elements satisfying

$$F^T(t)F(t) \le I, \quad F_i^T(t)F_i(t) \le I.$$

For system (1) with uncertainty (12), we can establish the following result by using Theorem 1 and the applying the S-procedure.

Theorem 2 For given constants τ_i, μ_i $(i = 1, \cdots, K)$, the system (1) subject to linear norm-bounded uncertainties (12) satisfying (2) is robustly stable if there exist matrices $P > 0$, $Z_i = Z_i^T > 0$, $Q_i = Q_i^T > 0$, N_i, M of appropriate dimensions and scalars $\varepsilon > 0$, $\varepsilon_i > 0\,(i = 1, \cdots, K)$ such that the following LMI holds

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ * & \Theta_{22} & 0 \\ * & * & \Theta_{33} \end{bmatrix} < 0, \qquad (13)$$

where

$$\Theta_{11} = \Xi_{11} + \mathrm{diag}\big(\varepsilon E^T E,\, \varepsilon_1 E_1^T E_1,\, \cdots,\, \varepsilon_K E_K^T E_K,\, 0\big), \quad \Theta_{12} = \Xi_{12},$$

$$\Theta_{13} = M[D \;\; D_1 \;\; \cdots \;\; D_k], \quad \Theta_{22} = \Xi_{22}, \quad \Theta_{33} = -\mathrm{diag}\big(\varepsilon I,\, \varepsilon_1 I,\, \cdots,\, \varepsilon_K I\big),$$

and Ξ_{11}, Ξ_{12}, Ξ_{22} and M are defined in (5).

Proof Replace A and A_i in (5) with $A+DF(t)E$ and $A_i+D_iF_i(t)E_i$ for $i=1,\cdots,K$, respectively, and multiply both sides of the resulting matrix by vectors x_j for $j=1,2,\cdots,2K+2$. Next, define $p=F(t)Ex_1$, $q_i=F_i(t)E_ix_{i+1}$, $i=1,\cdots,K$. Then, we have

$$\beta^T\begin{bmatrix}\Xi_{11} & \Xi_{12} & \Theta_{13}\\ * & \Xi_{22} & 0\\ * & * & 0\end{bmatrix}\beta<0 \tag{14}$$

for all admissible uncertainties $F(t)$, $F_i(t)(i=1,\cdots,K)$, where

$$\beta^T=\left[x_1^T,\,x_2^T,\,\cdots,\,x_{2K+2}^T,\,p^T,\,q_1^T,\,\cdots,\,q_K^T\right].$$

Since

$$F^T(t)F(t)\le I,\quad F_i^T(t)F_i(t)\le I,\quad i=1,\cdots,K,$$

it is obvious that the following inequalities hold for scalars $\varepsilon>0$, $\varepsilon_i>0\,(i=1,\cdots,K)$

$$\varepsilon p^Tp=\varepsilon x_1^TE^TF^T(t)F(t)Ex_1\le\varepsilon x_1^TE^TEx_1,\quad \varepsilon_iq_i^Tq_i\le\varepsilon_ix_{i+1}^TE_i^TE_ix_{i+1}. \tag{15}$$

Applying the S-procedure to (14) and inequalities (15) allow us to obtain (13). This completes the proof.

Similarly to Corollary 1, by choosing $Q_i\equiv0\,(i=1,\cdots,K)$ in Theorem 2, we can obtain the following result.

Corollary 2 For given constants $\tau_i\,(i=1,\cdots,K)$, the system (1) subject to linear norm-bounded uncertainties (12) satisfying (3) is robustly stable if there exist matrices $P>0$, $Z_i=Z_i^T>0$, N_i, M of appropriate dimensions and scalars $\varepsilon>0$, $\varepsilon_i>0\,(i=1,\cdots,K)$ such that the LMI (13) holds with $Q_i\equiv0\,(i=1,\cdots,K)$.

4 Conclusions

This paper presents some stability criteria for the uncertain system with multiple time-varying delays. The new integral-equality was developed to make the criteria less conservative, which is an improved version of the existing integral inequality. Our results are much less conservative and more general than relevant current results.

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不确定多时变线性系统的时滞依赖稳定性

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摘 要: 利用积分等式，本文研究了含范数有界不确定型的多时变线性系统的时滞依赖稳定性。所用积分等式通过自由权矩阵够造得到，比现有积分不等式更优，而自由项的选取依赖于Lyapunov-Krasovskii泛函。相比之前的一些结果，文中得到的稳定性保守性更低且更一般化。

关键词: Lyapunov-Krasovskii泛函；稳定性；时滞；积分等式；线性矩阵不等式(LMI)